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International Journal of Solids and Structures 43 (2006) 5100–5109

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

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Stress singularities resulting from various boundary conditions in angular corners of plates of arbitrary thickness in extension

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Received 14 April 2005; received in revised form 13 June 2005
Available online 18 August 2005

Abstract

The stress singularities in angular corners of plates of arbitrary thickness with various boundary conditions subjected to in-plane loading are studied within the first-order plate theory. By adapting an eigenfunction expansion approach a set of characteristic equations for determining the structure and orders of singularities of the stress resultants in the vicinity of the vertex is developed. The characteristic equations derived in this paper incorporate that obtained within the classical plane theory of elasticity (M.L. Williams' solution) and also describe the possible singular behaviour of the out-of-plane shear stress resultants induced by various boundary conditions.

Keywords: Stress singularities; Plates of arbitrary thickness in extension; Eigenvalue expansion approach; First-order plate theory

1. Introduction

Stress analysis based on the classical plane stress theory of elasticity occasionally leads to misleading results due, in part, to the fact that it is an approximate theory even when the plane stress equations are solved exactly. A case in point relates to an angular corner with the vertex angles between π and 2π. The lateral contraction of the plate at the vertex is unbounded. Near a vertex, the gradients of the in-plane stress components are very large. If the through-the-thickness contraction due to the Poisson effect is allowed to develop without resistance, which is the case in the plane stress theory of elasticity, then an

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0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2005.06.037

extremely large transverse shear strain arises. Associated with this strain, of course, is a transverse shear stress, which may be too large in magnitude to permit the basic assumption of the classical plane stress theory of elasticity (Yang and Freund, 1985).

Many papers have addressed stress singularities at angular corners or finite opening cracks. Williams and his co-workers first used the eigenfunction expansion approach to comprehensively investigate the corner stress singularities induced by various boundary conditions for isotropic and orthotropic plates within the generalized plane stress theory (Williams, 1952; Williams, 1957; Williams and Chapkis, 1958). Dempsey and Sinclair (1979) proposed a new form of Airy stress function to re-examine the stress singularities in isotropic elastic plates in extension. Hein and Erdogan (1971) and Bogy and Wang (1971) used the Mellin transformation to study the stress singularities in bi-material wedges, while Dempsey and Sinclair (1979) used an Airy stress function for the same purpose. Using a complex potential approach Carpenter (1985) examined the form of the eigenvector solution for a general corner or finite opening crack and developed an overdetermined collocation algorithm to calculate the coefficients associated with eigenvectors. Sinclair (1999) studied logarithmic stress singularities resulting from various boundary conditions in angular corners of plates in extension. In the frame of a high-order plate theory Huang (2004) studied corner singularity for a sector plate in extension by adapting an approach of Hartranft and Sih (1969) for three-dimensional problems and concluded that the characteristic equations for angular corners in extension are identical to those given by Williams for plane strain conditions. The listed above are only key references. No attempt has been made here to give a complete bibliography on the subject, or to analyse critically the works referred to, or to discuss the priority of the results mentioned.

The aim of this paper is to study the corner singularities for a sector plate within the first-order plate theory by using stress resultant and displacement functions (Kotousov and Wang, 2002a; Kotousov, 2004) and adapting the eigenfunction expansion approach of Williams (1952). The first-order plate theory, also known as the Kane and Mindlin theory, was first proposed in their work on high frequency extensional vibrations (Kane and Mindlin, 1956). The governing equations of this theory include the through-the-thickness stress components and retain the simplicity of a two-dimensional model. Recently, a number of three-dimensional analytical solutions have been developed within this theory (Kotousov and Wang, 2002a, 2002b; Kotousov, 2004, 2005). It was shown that these solutions mirror non-singular classical plane stress and plane strain solutions of the theory of elasticity as limiting cases of very thin and very thick plates, correspondingly. It also was found in a number of careful numerical studies that these solutions obtained within the first-order plate theory also agree well with results obtained using the three-dimensional finite element method (Chang et al., 2001; Berto et al., 2004).

2. Basic equations

For convenience, the basic equations of the first-order plate theory for extensional deformations will be summarized next. The three-dimensional displacements in this theory are taken in the form

$$u_x = u_x(x, y), \quad u_y = u_y(x, y), \quad u_z = \frac{z}{h} w(x, y), \tag{1}$$

which are also known as the Kane and Mindlin assumption and $2h$ is the plate thickness.

The stress resultants are defined by

$$(N_{xx}, N_{yy}, N_{zz}, N_{xy}) = \int_{-h}^{h} (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy})\, \mathrm{d}z, \tag{2a}$$

$$(R_x, R_y) = \int_{-h}^{h} (\tau_{zx}, \tau_{zy}) z\, \mathrm{d}z. \tag{2b}$$

It is seen that stress resultants N_{xx}, N_{yy} and N_{xy} are the usual forces per unit length, and N_{zz} is $2h$ times the average transverse normal stress. R_x and R_y are components of "pinching" shear, playing a role similar to that of the transverse shear force in the corresponding equilibrium equations of flexible plates.

The substituting Eq. (1) into the classical variational equation of the 3D theory of elasticity (Yu, 1995) and carrying out integration with respect to z over the thickness of the plate leads to the governing equations of the first-order plate theory (Kane and Mindlin, 1956). By introducing a stress resultant function Φ, similar to the Airy's stress function in the classical plane theory of elasticity, in polar co-ordinates

$$N_{rr} = \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{r}\frac{\partial \Phi}{\partial \phi}, \tag{3a}$$

$$N_{\phi\phi} = \frac{\partial^2 \Phi}{\partial r^2}, \tag{3b}$$

$$N_{r\phi} = -\frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \phi} + \frac{1}{r^2}\frac{\partial \Phi}{\partial \phi} \tag{3c}$$

and the governing equations of the first-order plate theory can be reduced to two equations (Yang and Freund, 1985), the first one is

$$\nabla^2 w - \frac{6(1+v)}{h^2} w = \frac{3v(1+v)}{h^2 E}\nabla^2 \Phi, \tag{4a}$$

which represents the equilibrium equation in the out-of-plane direction (z-direction, see Fig. 1). The second governing equation is

$$\nabla^4 \Phi = \frac{2vE}{1-v^2}\nabla^2 w, \tag{4b}$$

which is the strain compatibility equation. Here E and v are Young modulus and Poisson's ratio, respectively and ∇^2 is the two-dimensional Laplace operator.

Now the problem is fully determined by two governing equations for the stress function Φ (4a) and the out-of-plane function w (4b). This system can be decoupled and transformed into a single equation with respect to either Φ or w as

$$\nabla^4 w - \kappa^2 \nabla^2 w = 0, \tag{5a}$$

$$\nabla^6 \Phi - \kappa^2 \nabla^4 \Phi = 0, \tag{5b}$$

where $\kappa^2 = \frac{1}{h^2}\frac{6}{1-v}$.

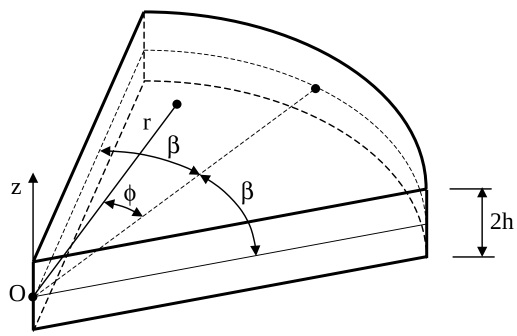

Fig. 1. Coordinate system for plate sector.

Further, using a harmonic displacement function $\Psi(\nabla^2\Psi = 0)$, the in-plane displacements in polar coordinates can be written as

$$\frac{2Eh}{1+v}u_r = -\frac{\partial\Phi}{\partial r} + \frac{1}{1+v}r\frac{\partial\Psi}{\partial\phi}, \tag{6a}$$

$$\frac{2Eh}{1+v}u_\phi = -\frac{1}{r}\frac{\partial\Phi}{\partial\phi} + \frac{1}{1+v}r^2\frac{\partial\Psi}{\partial r}, \tag{6b}$$

$$2vEw = (1-v^2)\nabla^2\Phi - \frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial\phi}\right). \tag{6c}$$

The harmonic displacement function Ψ can be determined in terms of the stress-resultant function Φ as

$$\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial\phi}\right) = \nabla^2\Phi - \frac{1}{\kappa^2}\nabla^4\Phi. \tag{6d}$$

Finally, the shear stress resultant can be expressed as

$$R_r = \frac{Eh^2}{3(1+v)}\frac{\partial w}{\partial r}, \tag{7a}$$

$$R_\phi = \frac{Eh^2}{3(1+v)}\frac{1}{r}\frac{\partial w}{\partial\phi}, \tag{7b}$$

and the normal out-of-plane stress resultant

$$N_{zz} = 2vN + 2Ew, \tag{8}$$

where

$$N = (N_{rr} + N_{\phi\phi})/2.$$

3. Approach

Consider a plane angular sector with the vertex angle 2β and thickness $2h$ as shown in Fig. 1. Following to the classical eigenfunction expansion approach developed by Williams (1952), let us assume solution for w and Φ in the form

$$w(r,\phi) = \sum_{k=0}^{\infty} w_{\mu+2k}(r,\phi), \tag{9a}$$

$$\Phi(r,\phi) = \sum_{k=0}^{\infty} \Phi_{\mu+2k}(r,\phi), \tag{9b}$$

$$\Psi(r,\phi) = \sum_{k=0}^{\infty} \Psi_{\mu+2k}(r,\phi), \tag{9c}$$

where

$$\begin{aligned} w_\lambda(r,\phi) = {} & A_k\langle\alpha r^{\lambda-2} - I_{\lambda-2}(\kappa r)\rangle\cos(\lambda-2)\phi + B_k I_\lambda(\kappa r)\cos\lambda\phi \\ & + \overline{A}_k\langle\alpha r^{\lambda-2} - I_{\lambda-2}(\kappa r)\rangle\sin(\lambda-2)\phi + \overline{B}_k I_\lambda(\kappa r)\sin\lambda\phi, \end{aligned} \tag{10a}$$

$$\begin{aligned}\Phi_\lambda(r,\phi) = {} & A_k \frac{2\nu E}{(1-\nu^2)\kappa^2}\langle \alpha r^{\lambda-2} - I_{\lambda-2}(\kappa r)\rangle \cos(\lambda-2)\phi - A_k \frac{\alpha E}{2\nu(\lambda-1)} r^\lambda \cos(\lambda-2)\phi \\ & + \overline{A}_k \frac{2\nu E}{(1-\nu^2)\kappa^2}\langle \alpha r^{\lambda-2} - I_{\lambda-2}(\kappa r)\rangle \sin(\lambda-2)\phi - \overline{A}_k \frac{\alpha E}{2\nu(\lambda-1)} r^\lambda \sin(\lambda-2)\phi \\ & + B_k \frac{2\nu E}{(1-\nu^2)\kappa^2} I_\lambda(\kappa r)\cos\lambda\phi + \overline{B}_k \frac{2\nu E}{(1-\nu^2)\kappa^2} I_\lambda(\kappa r)\sin\lambda\phi, \\ & + C_k r^\lambda \cos\lambda\phi + \overline{C}_k r^\lambda \sin\lambda\phi, \end{aligned} \tag{10b}$$

and

$$\Psi_\lambda(r,\phi) = \overline{A}_k \frac{2\alpha E}{\nu(\lambda-1)(\lambda-2)} r^{\lambda-2}\cos(\lambda-2)\phi - A_k \frac{2\alpha E}{\nu(\lambda-1)(\lambda-2)} r^{\lambda-2}\sin(\lambda-2)\phi, \tag{10c}$$

where A_k, $\overline{A}_k$, B_k, $\overline{B}_k$, C_k and $\overline{C}_k$ are constants to be found from boundary conditions.

Each term in these functions w_λ, Φ_λ and Ψ_λ represents a linear independent solution of the governing Eqs. (4) or (5) and (6d) having the same asymptotic behaviour at $r \to 0$ and resulting in the following general asymptotic behaviour of these functions:

$$\Phi(r,\phi) \sim r^\lambda, \quad w(r,\phi) \sim r^\lambda \text{ and } \Psi_\lambda(r,\phi) \sim r^{\lambda-2}. \tag{11}$$

In Eq. (10) $\alpha = \frac{1}{\Gamma(\lambda-1)}(\kappa/2)^{\lambda-2}$ leading to the asymptotic behavior (11), $\Gamma(\cdot)$ is the Gamma function, $I_\lambda(\cdot)$ is the modified Bessel function of the first kind and λ is a dummy variable.

It is noted that the odd terms (such as $\mu + 2k + 1$) in the expansion of functions w, Φ and Ψ will not generate any additional solution; therefore, they are not considered in the representation of these functions (9).

Boundary conditions along radial edges require:

For free–free edges:

$$N_{\phi\phi} = 0, \quad N_{r\phi} = 0 \quad \text{and} \quad R_\phi = 0 \text{ at } \phi = \pm\beta, \tag{12a}$$

or using (3)

$$\frac{\partial \Phi}{\partial r} = 0, \quad \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial \Phi}{\partial \phi} = 0 \quad \text{and} \quad \frac{\partial w}{\partial \phi} = 0 \text{ at } \phi = \pm\beta, \tag{12b}$$

respectively.

For clamped–clamped edges:

$$u_r = 0, \quad u_\phi = 0 \quad \text{and} \quad w = 0 \text{ at } \phi = \pm\beta, \tag{13a}$$

first two conditions can be rewritten using (6) as

$$-\frac{\partial \Phi}{\partial r} + \frac{1}{1+\nu} r \frac{\partial \Psi}{\partial \phi} = 0, \tag{13b}$$

$$-\frac{1}{r}\frac{\partial \Phi}{\partial \phi} + \frac{1}{1+\nu} r^2 \frac{\partial \Psi}{\partial r} = 0. \tag{13c}$$

For clamped–free edges:

$$u_r = 0, \quad u_\phi = 0 \quad \text{and} \quad w = 0 \text{ at } \phi = \beta, \tag{14a}$$

$$N_{\phi\phi} = 0, \quad N_{r\phi} = 0 \quad \text{and} \quad R_\phi = 0 \text{ at } \phi = -\beta, \tag{14b}$$

which also can be rewritten in terms of functions $w(r,\phi)$, $\Phi(r,\phi)$ and $\Psi(r,\phi)$ similar to the above Eqs. (12b), (13b) and (13c).

By using the well-known representation of Bessel functions into the power series (Gradshteyn and Ryzhik, 1981)

$$I_\lambda(\omega) = \sum_{m=0}^{\infty} \frac{(\omega/2)^{2m+\lambda}}{m!\Gamma(m+\lambda+1)},$$

after substituting of Eqs. (9) and (10) into the boundary conditions (12)–(14) and requiring that the coefficients of r with different orders equal to zero one gets a system of recurrent equations for the coefficients A_k, $\overline{A}_k$, B_k, $\overline{B}_k$, C_k and $\overline{C}_k$.

To investigate the structure and orders of singularities at the angular corner within the first-order plate theory, one needs the asymptotic form of the stress resultants. The singular behaviour of the stress resultants is determined from w_μ, Φ_μ and Ψ_μ (at $k = 0$) corresponding to the lowest order of r, therefore, the solution for $k > 0$ in (9) will not be considered here.

Substituting w_μ, Φ_μ and Ψ_μ given by Eqs. (10) into the boundary conditions (12)–(14) one gets a system of six simultaneous equations for six unknown constants A_0, $\overline{A}_0$, B_0, $\overline{B}_0$, C_0 and $\overline{C}_0$. These equations are homogeneous and a meaningful solution can be obtained if the determinants of the coefficient matrices are each equal to zero. Then, the characteristic equation for μ can be established. After some algebraic manipulations with the coefficients of the system and Gamma function, the characteristic equations can be written in the following simple form:

For free–free edges:

$$\Delta_S(\mu) = C_1\langle \sin 2(\mu-1)\beta + (\mu-1)\sin 2\beta\rangle \sin \mu\beta, \tag{15a}$$

$$\Delta_A(\mu) = C_1\langle \sin 2(\mu-1)\beta - (\mu-1)\sin 2\beta\rangle \cos \mu\beta. \tag{15b}$$

For clamped–clamped edges:

$$\Delta_S(\mu) = C_2\left\langle \sin 2(\mu-1)\beta + \frac{\mu-1}{3-4v}\sin 2\beta\right\rangle \cos \mu\beta, \tag{16a}$$

$$\Delta_A(\mu) = C_2\left\langle \sin 2(\mu-1)\beta + \frac{\mu-1}{3-4v}\sin 2\beta\right\rangle \sin \mu\beta, \tag{16b}$$

where $\Delta_S(\mu)$ and $\Delta_A(\mu)$ correspond to the symmetric and anti-symmetric loading cases, respectively.

For clamped–free edges:

$$\Delta(\mu) = C_3\left\langle \sin^2 2(\mu-1)\beta - \frac{4(1-v)^2}{3-4v} + \frac{(\mu-1)^2\sin 2\beta}{3-4v}\right\rangle \cos 2\mu\beta, \tag{17}$$

where C_1, C_2 and C are functions of material constants and μ such that at $\mu > 0$ $C \neq 0$ except when $\mu = 1$.

The details of the derivation of Eqs. (15)–(17) will be omitted. Note that, Eq. (16a) was also obtained in (Huang, 2004) using a different approach based on the expansion of the displacement functions; however the additional term $\cos(\mu\beta)$ was discarded from further considerations as not giving any singularities at the vertex of the wedge.

4. Analysis

Expressions in $\langle\cdot\rangle$ brackets in Eqs. (15)–(17) are exactly the same as given by Williams (1952) for the corresponding plane strain problem of the wedge. It was shown in work by Williams (1952) and many other papers that the roots of these equations such that Re $\mu > 1$ produce an admissible solution for this problem. Furthermore, a value of μ such that $1 < Re\,\mu < 2$ produces unbounded in-plane stress resultant behaviour at the vertex of the wedge. Nevertheless, no singularities occur for the shear stress resultants R_r and R_ϕ, which are, in this case, proportional to $r^{\mu-1}$.

This type of singularities was investigated exhaustively in many papers and will not be further analysed here. The additional terms in the characteristic equations $\cos(\mu\beta)$, $\sin(\mu\beta)$ and $\cos(2\mu\beta)$ can be substituted back into the system of homogeneous equations for the determining the coefficients A_0, $\overline{A}_0$, B_0, $\overline{B}_0$, C_0 and $\overline{C}_0$ to obtain a solution for another type of singular behaviour.

The out-of-plane displacement function w and the stress resultant function Φ are found to be

For free–free edges:

$$w_\mu(r,\phi) = B_0 I_\mu(\kappa r)\cos\mu\phi, \tag{18a}$$

$$\Phi_\mu(r,\phi) = B_0\frac{2vE}{(1-v^2)\kappa^2}\left\langle I_\mu(\kappa r) - \frac{\kappa^\mu}{2^\mu}\frac{r^\mu}{\Gamma(\mu+1)}\right\rangle\cos\mu\phi, \tag{18b}$$

for symmetric loading corresponding to the roots of $\sin(\mu\beta) = 0$, and

$$w_\mu(r,\phi) = \overline{B}_0 I_\mu(\kappa r)\sin\mu\phi, \tag{18c}$$

$$\Phi_\mu(r,\phi) = \overline{B}_0\frac{2vE}{(1-v^2)\kappa^2}\left\langle I_\mu(\kappa r) - \frac{\kappa^\mu}{2^\mu}\frac{r^\mu}{\Gamma(\mu+1)}\right\rangle\sin\mu\phi \tag{18d}$$

for anti-symmetric loading corresponding to the roots of $\cos(\mu\beta) = 0$.

For clamped–clamped edges:

$$w_\mu(r,\phi) = \overline{B}_0 I_\mu(\kappa r)\cos\mu\phi, \tag{19a}$$

$$\Phi_\mu(r,\phi) = \overline{B}_0\frac{2vE}{(1-v^2)\kappa^2}\left\langle I_\mu(\kappa r) - \frac{\kappa^\mu}{2^\mu}\frac{r^\mu}{\Gamma(\mu+1)}\right\rangle\cos\mu\phi, \tag{19b}$$

for symmetric loading corresponding to the roots of $\cos(\mu\beta) = 0$, and

$$w_\mu(r,\phi) = B_0 I_\mu(\kappa r)\sin\mu\phi, \tag{19c}$$

$$\Phi_\mu(r,\phi) = B_0\frac{2vE}{(1-v^2)\kappa^2}\left\langle I_\mu(\kappa r) - \frac{\kappa^\mu}{2^\mu}\frac{r^\mu}{\Gamma(\mu+1)}\right\rangle\sin\mu\phi \tag{19d}$$

for anti-symmetric loading corresponding to the roots of $\sin(\mu\beta) = 0$.

For clamped–free edges:

$$w_\mu(r,\phi) = B_0 I_\mu(\kappa r)(\sin\mu\phi - \mathrm{ctg}(\mu\beta)\cos\mu\phi), \tag{20a}$$

$$\Phi_\mu(r,\phi) = B_0\frac{2vE}{(1-v^2)\kappa^2}\left\langle I_\mu(\kappa r) - \frac{\kappa^\mu}{2^\mu}\frac{r^\mu}{\Gamma(\mu+1)}\right\rangle(\sin\mu\phi - \mathrm{ctg}(\mu\beta)\cos\mu\phi) \tag{20b}$$

corresponding to the roots of $\cos(2\mu\beta) = 0$.

For all cases the displacement function Ψ_μ is zero.

Because solutions (18)–(20) were derived from the general solution (10) they obviously satisfy the governing Eqs. (4), (5) and (6d). It is readily seen that the boundary conditions (12)–(14) for w and R_ϕ are also satisfied exactly. The stress resultant function $\Phi(r,\phi)$ and the out-of-plane displacement function $w(r,\phi)$ in the solutions (18)–(20) have the asymptotic behaviour as

$$\Phi_\mu(r,\phi) \sim r^{\mu+2} \quad \text{and} \quad w_\mu(r,\phi) \sim r^\mu, \tag{21}$$

leading to the following asymptotic behaviour of the stress resultants, out-of-plane displacement function and in-plane displacements:

$$R_r \quad \text{and} \quad R_\phi \sim r^{\mu-1}, \tag{22a}$$

$$w, N_{rr}, N_{r\phi}, N_{\phi\phi} \quad \text{and} \quad N_{zz} \sim r^{\mu}, \tag{22b}$$

$$u_r \quad \text{and} \quad u_\phi \sim r^{\mu+1}. \tag{22c}$$

It means that the boundary conditions (12)–(14) for the stress resultants $N_{\phi\phi}$, $N_{r\phi}$ and the in-plane displacement components u_r and u_ϕ in the above solutions (18)–(20) are also satisfied asymptotically in terms of $r^{\mu-2}$ and $r^{\mu-1}$, respectively. However, these solutions produce additional terms of O(r^{μ}) order for the stress resultants and O($r^{\mu+1}$) order for the in-plane displacements, which can be negated by the higher order terms ($k > 0$) in the expansion of functions w, Φ and Ψ (Eq. (9)) leading to the system of recurrent equations as described above.

Now we investigate the roots of the additional terms in the characteristic equations. The positive roots ($\mu > 0$) of $\sin\mu\beta$, $\cos\mu\beta$ and $\cos(2\mu\beta)$ give an admissible solution for this problem, as the elastic energy will stay finite at the vertex sector (Yu, 1995) with the stress resultants having the asymptotic behaviour (22).

The positive roots of $\sin\mu\beta$, $\cos\mu\beta$ and $\cos 2\mu\beta$ are

$$\mu = (n+1)\pi/\beta, \tag{23a}$$

$$\mu = (2n+1)\pi/(2\beta), \tag{23b}$$

and

$$\mu = (2n+1)\pi/(4\beta), \tag{23c}$$

respectively, with $n = 0, 1, 2\ldots$

The unbounded behaviour of the out-of-plane shear stress resultants R_r and R_ϕ at the vertex sector will occur when $0 < \mu < 1$, which is possible for $n = 0$ for the anti-symmetric mode of loading of the wedge with free–free boundary conditions and symmetric mode of loading for clamped–clamped boundary conditions when $\beta > \pi/2$ (or the vertex angle larger than π). The characteristic equation and roots of this equation are defined as

$$\mu = \pi/(2\beta). \tag{24}$$

For clamped–free boundary condition we have two different equations giving an admissible singular solution for the problem under consideration generated by $n = 0$ and by $n = 1$ in (23c) for the vertex angle between 0 and 2π:

$$\mu = \pi/(4\beta), \tag{25a}$$

and

$$\mu = (3\pi)/(4\beta). \tag{25b}$$

These Eqs. (25a) and (25b) result into the singular behaviour of the out-of-plane shear stress resultants when $\beta > \pi/4$ and $\beta > 3\pi/4$, respectively. It means that two different modes of singular behaviour of the shear stress resultants can take place at $\beta > 3\pi/4$ and only one mode for $\pi/4 < \beta < 3\pi/4$. Meanwhile no singularities occur for the in-plane stress resultants N_{rr}, $N_{\phi\phi}$ and $N_{r\phi}$ in this case.

5. Conclusion

By adapting the eigenfunction expansion approach (Williams, 1952) and using the stress resultant and displacement functions for the first order plate theory (Kotousov, 2005) a set of characteristic equations for determining the structure and orders of singularities in angular corner of arbitrary thickness with various boundary conditions is developed. The characteristic equations derived in this paper differ from Williams' results and are not related to the thickness of the plate.

Similar to the Williams' solution obtained within the plane theory of elasticity, for free–free and clamped–clamped boundary conditions the characteristic equations can be represented as a product of symmetric and anti-symmetric parts corresponding to the symmetric and anti-symmetric loading cases of the wedge, respectively. Further, all characteristic equations can be decomposed into the characteristic equations describing in-plane singular behaviour and additional equations, which can also produce an admissible singular solution for the problem under consideration. The first type of singularities does not lead to unbounded out-of-plane shear stress resultants. In its turn, for the second type of singularities the in-plane stress resultants stay finite while the out-of-plane shear stress resultants are unbounded. It was shown in this paper that the second type of singularities is possible for the anti-symmetric mode of loading for free–free boundary conditions and the symmetric mode of loading for clamped–clamped boundary conditions with the vertex angle larger than π. In the case of clamped–free boundary conditions the shear stress resultant singular behaviour is possible for the vertex angle larger than $\pi/2$. In a broad sense, it means that the singular fields in the vicinity of the vertex will be controlled by two or three (for clamped–free boundary conditions and the vertex angle larger than $6\pi/4$) uncoupled singular modes, the in-plane mode producing singular behaviour of the in-plane stress resultants and the out-of-plane mode producing a singular behaviour of the out-of-plane shear stress resultants. Each of these modes, obviously, can be characterized by a generalized stress intensity factor. This result represents a major finding of this paper.

Acknowledgement

This work reported herein was supported by the Australian Research Council, through research Grant No. DP0557124. The support is gratefully acknowledged.

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